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Augmented superfield approach to exact nilpotent symmetries for matter fields in non-Abelian theory

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Abstract. We derive the nilpotent (anti-) BRST symmetry transformations for the Dirac (matter) fields of an interacting four (3+1)-dimensional 1-form non-Abelian gauge theory by applying the theoretical arsenal of augmented superfield formalism where (i) the horizontality condition, and (ii) the equality of a gauge invariant quantity, on the six (4,2)-dimensional supermanifold, are exploited together. The above supermanifold is parameterized by four bosonic spacetime coordinates x^μ (with $\mu = 0, 1, 2, 3$) and a couple of Grassmannian variables θ and $\bar{\theta}$. The on-shell nilpotent BRST symmetry transformations for all the fields of the theory are derived by considering the chiral superfields on the five (4,1)-dimensional super sub-manifold and the off-shell nilpotent symmetry transformations emerge from the consideration of the general superfields on the full six (4,2)-dimensional supermanifold. Geometrical interpretations for all the above nilpotent symmetry transformations are also discussed within the framework of augmented superfield formalism.

1 Introduction

The usual superfield approach [1–6] to Becchi–Rouet–Stora–Tyutin (BRST) formalism provides the geometrical origin and interpretations for some of the abstract mathematical properties associated with the nilpotent and conserved (anti-) BRST charges (and the nilpotent symmetry transformations they generate) for the gauge and (anti-) ghost fields of the p-form ($p = 1, 2, 3 \ldots$) interacting gauge theories. This approach, however, does not shed any light on the (anti-) BRST symmetry transformations (and the corresponding generators) for the matter fields that have interactions with the p-form gauge fields of the above interacting gauge theories. It has been an interesting and challenging problem to derive these symmetry transformations for the matter fields within the framework of superfield formalism without spoiling the beauty of the geometrical interpretations (for the conserved and nilpotent (anti-) BRST charges, the corresponding nilpotent transformations for the gauge and (anti-) ghost fields, their associated key properties, etc.), which emerge from the horizontality condition alone.

To elaborate a bit on the usual superfield approach to BRST formalism (endowed with the theoretical arsenal of the horizontality condition alone), it will be noted that, for the D-dimensional Abelian p-form gauge theories, a $(p+1)$-form super curvature[1] $\tilde{F}^{(p+1)} = \tilde{d}\tilde{A}^{(p)}$ is constructed from the super exterior derivative $\tilde{d} = dx^\mu \partial_\mu + d\theta\partial_\theta + d\bar{\theta}\partial_{\bar{\theta}}$ (with $\tilde{d}^2 = 0$) and the super p-form connection $\tilde{A}^{(p)}$ on a $(D, 2)$-dimensional supermanifold parameterized by the D-number of bosonic spacetime variables x^μ (with $\mu = 0, 1, 2 \ldots D-1$) and a couple of Grassmannian variables θ and $\bar{\theta}$ (with $\theta^2 = \bar{\theta}^2 = 0, \theta\bar{\theta} + \bar{\theta}\theta = 0$). This is subsequently equated with the D-dimensional ordinary $(p+1)$-form curvature $F^{(p+1)} = dA^{(p)}$ constructed from the ordinary exterior derivative $d = dx^\mu \partial_\mu$ (with $d^2 = 0$) and the ordinary p-from connection $A^{(p)} = \frac{1}{p!}(dx^{\mu_1} \wedge dx^{\mu_2} \ldots \wedge dx^{\mu_p}) A_{\mu_1, \mu_2 \ldots \mu_p}$ due to the horizontality condition (which has been christened as the soul-flatness condition in [7]). This leads to the derivation of the nilpotent and anti-commuting (anti-) BRST symmetry transformations for only the gauge and (anti-) ghost fields of the D-dimensional Lagrangian density of an (anti-) BRST invariant p-form Abelian gauge theory.

In a recent set of papers [8–13], the above horizontality condition was consistently extended[2] by requiring the equality of (i) the conserved currents/charges, and

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[1] For the 1-form non-Abelian gauge theory, the super curvature 2-form $\tilde{F}^{(2)} = \tilde{d}\tilde{A}^{(1)} + \mathrm{i}\tilde{A}^{(1)} \wedge A^{(1)}$ is equated with the ordinary 2-form $F^{(2)} = dA^{(1)} + \mathrm{i}A^{(1)} \wedge A^{(1)}$ due to the horizontality condition that leads to the derivation of the nilpotent (anti-) BRST symmetry transformations for the gauge and (anti-)ghost fields (see, e.g., [3] for details).

[2] These extended versions have been christened as the augmented superfield formulation.

(ii) the gauge (i.e., BRST) invariant quantities that owe their origin to the (super) covariant derivatives on the appropriate supermanifolds, so that the nilpotent (anti-) BRST symmetry transformations for the matter (or its analogue) fields can be derived in a logical fashion. The former restriction leads to the consistent derivation of the nilpotent symmetry transformations for the matter fields, whereas the latter restriction yields the mathematically exact nilpotent symmetry transformations for the matter fields. Both these extensions of the usual superfield formalism have their own merits. For instance, the former is applicable to any reparameterization and/or gauge invariant theories where the covariant derivatives do not play such an important role (e.g., the system of (super) relativistic particles [12, 13]). The latter crucially depends on the existence of covariant derivatives in the theory (e.g., the $U(1)$ and $SU(N)$ gauge invariant theories [11, 14, 15]).

One of the key points of the above extensions is the fact that the geometrical interpretations for the (anti-) BRST charges (and the corresponding nilpotent symmetry transformations generated by them) remain exactly the same as in the usual superfield approach to BRST formalism [1–7]. For instance, the above (anti-) BRST charges turn out to be the translational generators (i.e., $(\mathrm{Lim}_{\bar\theta\to 0}(\partial/\partial\theta))$ $\mathrm{Lim}_{\theta\to 0}(\partial/\partial\bar\theta))$ along the Grassmannian directions of the $(D,2)$-dimensional supermanifold. Their nilpotency property is found to be encoded in the two successive translations (i.e., $(\partial/\partial\theta)^2=0, (\partial/\partial\bar\theta)^2=0$) of the superfields along any particular Grassmannian direction of the suitably chosen $(D,2)$-dimensional supermanifold. The anticommutativity property of the BRST and anti-BRST charges are found to be linked with such a kind of property (i.e., $(\partial/\partial\theta)(\partial/\partial\bar\theta)+(\partial/\partial\bar\theta)(\partial/\partial\theta)=0$) associated with the translational generators along the Grassmannian directions. Finally, the internal BRST and anti-BRST symmetry transformations for the local D-dimensional ordinary fields are geometrically equivalent to the translations of the corresponding superfields along the Grassmannian directions of the $(D,2)$-dimensional supermanifold.

In our present investigation, we exploit the latter extension of the usual superfield approach to BRST formalism to obtain the mathematically exact nilpotent (anti-) BRST symmetry transformations for the matter (Dirac) fields of the physical four (3+1)-dimensional (4D) non-Abelian gauge theory. First, as a warm-up exercise, we derive the on-shell nilpotent BRST symmetry transformations for the matter as well as gauge and (anti-) ghost fields by invoking the definition of the chiral superfields on the five (4,1)-dimensional chiral super sub-manifold of the general (4,2)-dimensional supermanifold. This exercise also provides, in a subtle way, the logical reason behind the non-existence of the on-shell nilpotent anti-BRST symmetry transformations for a certain specific set of Lagrangian densities of the interacting 4D non-Abelian gauge theory. Later on, we derive the off-shell nilpotent (anti-) BRST symmetry transformations for all the fields of the 4D non-Abelian gauge theory by considering the general superfields on the general six (4,2)-dimensional supermanifold. For this purpose, we tap the potential and power of the horizontality condition as well as the additional gauge-invariant restriction on the above supermanifold. We demonstrate that the above restrictions are beautifully intertwined and possess a common mathematical origin.

Our present endeavour is essential basically on three counts. First, it has been a challenging problem to derive the nilpotent (anti-) BRST symmetry transformations for the matter fields of any arbitrary gauge theory within the framework of the superfield approach to BRST formalism without having any conflict with the key results and observations of the usual superfield formalism [1–7]. Second, to check the validity of the gauge-invariant restriction in yielding mathematically exact nilpotent (anti-) BRST transformations for the matter fields of an interacting 4D non-Abelian $SU(N)$ gauge theory, which was found to be true for the interacting 4D Abelian $U(1)$ gauge theory[3] [14, 15]. Finally, the ideas proposed in our present investigation might turn out to be useful in the derivation of the nilpotent (anti-) BRST symmetry transformations for the reparameterization invariant theories of gravitation, where the covariant derivatives play very important roles in generating the interaction terms. It is well-known that some of the key features of non-Abelian gauge theories are very closely connected to a few central ideas in the theory of gravitation (see, e.g., [16] for details).

The contents of our present paper are organized as follows. To set up the notations and conventions, in Sect. 2 we discuss the bare essentials of the (anti-) BRST symmetry transformations for the four (3+1)-dimensional interacting non-Abelian gauge theory. Section 3 is devoted to the derivation of the on-shell nilpotent BRST symmetry transformations in the augmented superfield formulation, where only the chiral superfields are considered on the five (4,1)-dimensional super sub-manifold of the general six (4,2)-dimensional supermanifold. The off shell nilpotent (anti-) BRST symmetry transformations for all the fields of the Lagrangian densities are derived in Sect. 4, where a general set of superfields is considered on the six (4,2)-dimensional supermanifold. Finally, in Sect. 5, we make some concluding remarks and point out a few future directions for further investigations.

2 Nilpotent symmetry transformations in Lagrangian formalism: a brief sketch

We begin with the BRST invariant Lagrangian density of the physical four (3+1)-dimensional non-Abelian 1-form (i.e., $A^{(1)}=dx^\mu A_\mu$) interacting gauge theory, in the Feynman gauge, as (see, e.g., [16, 17])

$$\mathcal{L}_b=-\frac{1}{4}F_{\mu\nu}\cdot F^{\mu\nu}+\bar\psi(\mathrm{i}\gamma^\mu D_\mu-m)\psi+B\cdot(\partial_\mu A^\mu)+\frac{1}{2}B\cdot B-\mathrm{i}\partial_\mu\bar C\cdot D^\mu C\,, \tag{1}$$

[3] It is self-evident that the latter gauge theory (i.e., Abelian) is the limiting case of the former (i.e., non-Abelian) theory that is theoretically more general in nature.

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i} A_\mu \times A_\nu$ is the field strength tensor for the group valued non-Abelian gauge potential $A_\mu \equiv A^a_\mu T^a$ with the group generators T's obeying the algebra $[T^a, T^b] = f^{abc} T^c$. The structure constant f^{abc} can be chosen to be totally anti-symmetric in the group indices a, b and c for a semi-simple Lie group [16]. The covariant derivatives $D_\mu \psi = \partial_\mu \psi + \mathrm{i} A^a_\mu \, T^a \psi$ and $D_\mu C^a = \partial_\mu C^a + \mathrm{i} f^{abc} A^b_\mu C^c \equiv \partial_\mu C^a + \mathrm{i}(A_\mu \times C)^a$ are defined[4] on the matter (quark) field ψ and ghost field C^a, such that $[D_\mu, D_\nu]\psi = \mathrm{i} F_{\mu\nu} \psi$ and $([D_\mu, D_\nu])C^a = \mathrm{i}(F_{\mu\nu} \times C)^a$. It will be noted that these definitions for $F_{\mu\nu}$ agree with the Maurer–Cartan equation $F^{(2)} = dA^{(1)} + \mathrm{i} A^{(1)} \wedge A^{(1)} \equiv \frac{1}{2!}(dx^\mu \wedge dx^\nu) F_{\mu\nu}$ that defines the 2-form $F^{(2)}$ which, ultimately, leads to the derivation of $F_{\mu\nu}$. In the above equation (1), B^a are the Nakanishi–Lautrup auxiliary fields, and the anti-commuting (i.e., $(C^a)^2 = 0 = (\bar{C}^a)^2, C^a \bar{C}^b + \bar{C}^b C^a = 0$) (anti-) ghost fields $(\bar{C}^a) C^a$ are required for the proof of unitarity in the 1-form non-Abelian gauge theory[5]. Furthermore, the γ's are the usual 4×4 Dirac matrices in the physical four (3+1)-dimensional Minkowski space.

The above Lagrangian density (1) respects the following off-shell nilpotent ($s_b^2 = 0$) BRST symmetry transformations (s_b) [16, 17]

$$
\begin{aligned}
&s_b A_\mu = D_\mu C, \quad s_b C = -\frac{\mathrm{i}}{2}(C \times C), \quad s_b \bar{C} = \mathrm{i} B, \\
&s_b B = 0, \quad s_b \psi = -\mathrm{i}(C \cdot T)\psi, \quad s_b \bar{\psi} = -\mathrm{i}\bar{\psi}(C \cdot T), \\
&s_b F_{\mu\nu} = \mathrm{i}(F_{\mu\nu} \times C) \,. \qquad (2)
\end{aligned}
$$

It will be noted that the kinetic energy term $\left(-\frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu}\right)$ remains invariant under the BRST transformations (i.e., $s_b(-\frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu}) = -\frac{i}{2} f^{abc} F^a_{\mu\nu} F^{\mu\nu b} C^c = 0$) because of the totally anti-symmetric property of the structure constants f^{abc}. The on-shell ($\partial_\mu D^\mu C = 0$) nilpotent ($\tilde{s}_b^2 = 0$) version of the above nilpotent symmetry transformations (i.e., $\tilde{s}_b$), are

$$
\begin{aligned}
&\tilde{s}_b A_\mu = D_\mu C, \quad \tilde{s}_b C = -\frac{\mathrm{i}}{2}(C \times C), \quad \tilde{s}_b \bar{C} = -\mathrm{i}(\partial_\mu A^\mu), \\
&\tilde{s}_b \psi = -\mathrm{i}(C \cdot T)\psi, \quad \tilde{s}_b \bar{\psi} = -\mathrm{i}\bar{\psi}(C \cdot T) \,, \\
&\tilde{s}_b F_{\mu\nu} = \mathrm{i}(F_{\mu\nu} \times C) \,, \qquad (3)
\end{aligned}
$$

under which the following Lagrangian density

$$
\begin{aligned}
\mathcal{L}_b^{(0)} = &-\frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu} + \bar{\psi}\,(\mathrm{i}\gamma^\mu D_\mu - m)\,\psi \\
&-\frac{1}{2}\,(\partial_\mu A^\mu) \cdot (\partial_\rho A^\rho) - \mathrm{i}\partial_\mu \bar{C} \cdot D^\mu C \,, \qquad (4)
\end{aligned}
$$

changes to a total derivative (i.e., $\tilde{s}_b \mathcal{L}_b^{(0)} = -\partial_\mu[(\partial_\rho A^\rho) \cdot D^\mu C]$).

The following off-shell nilpotent ($s_{ab}^2 = 0$) version of the anti-BRST (s_{ab}) transformations

$$
\begin{aligned}
&s_{ab} A_\mu = D_\mu \bar{C}, \quad s_{ab} \bar{C} = -\frac{\mathrm{i}}{2}(\bar{C} \times \bar{C}), \quad s_{ab} C = \mathrm{i}\bar{B} \,, \\
&s_{ab} B = \mathrm{i}(B \times \bar{C}), \quad s_{ab} F_{\mu\nu} = \mathrm{i}(F_{\mu\nu} \times \bar{C}), \quad s_{ab} \bar{B} = 0 \,, \\
&s_{ab} \psi = -\mathrm{i}(\bar{C} \cdot T)\psi, \quad s_{ab} \bar{\psi} = -\mathrm{i}\bar{\psi}(\bar{C} \cdot T) \,, \qquad (5)
\end{aligned}
$$

are the symmetry transformations for the following equivalent Lagrangians

$$
\begin{aligned}
\mathcal{L}_{\bar{B}}^{(1)} = &-\frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu} + \bar{\psi}\,(\mathrm{i}\gamma^\mu D_\mu - m)\,\psi + B \cdot (\partial_\mu A^\mu) \\
&+\frac{1}{2}(B \cdot B + \bar{B} \cdot \bar{B}) - \mathrm{i}\partial_\mu \bar{C} \cdot D^\mu C \,, \qquad (6) \\
\mathcal{L}_{\bar{B}}^{(2)} = &-\frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu} + \bar{\psi}\,(\mathrm{i}\gamma^\mu D_\mu - m)\,\psi - \bar{B} \cdot (\partial_\mu A^\mu) \\
&+\frac{1}{2}(B \cdot B + \bar{B} \cdot \bar{B}) - \mathrm{i} D_\mu \bar{C} \cdot \partial^\mu C \,, \qquad (7)
\end{aligned}
$$

where another auxiliary field $\bar{B}$ has been introduced with the restriction $B + \bar{B} = -(C \times \bar{C})$ (see, e.g. [19]). It can be checked that the anti-commutativity property ($s_b s_{ab} + s_{ab} s_b = 0$) is true for any arbitrary field of the above Lagrangian densities. For the proof of this statement, one should also take into account $s_b \bar{B} = i(\bar{B} \times C)$, which is not mentioned in (2). We emphasize that the on-shell version of the anti-BRST symmetry transformations does not exist for any of the above cited Lagrangian densities (see, e.g. [20]).

All the above nilpotent symmetry transformations can be succinctly expressed in terms of the conserved and off-shell nilpotent (anti-) BRST charges Q_r and on-shell nilpotent BRST charge $\tilde{Q}_b$, as

$$
s_r \varPhi = -\mathrm{i}[\varPhi, Q_r]_\pm, \quad r = b, ab, \quad \tilde{s}_b \tilde{\varPhi} = -\mathrm{i}[\tilde{\varPhi}, \tilde{Q}_b]_\pm \,, \qquad (8)
$$

where the (+)− signs, as the subscripts, on the square brackets stand for the (anti-) commutator for the generic field $\varPhi = A_\mu, C, \bar{C}, \psi, \bar{\psi}, B, \bar{B}$ and $\tilde{\varPhi} = A_\mu, C, \bar{C}, \psi, \bar{\psi}$ of the above appropriate Lagrangian densities being (fermionic) bosonic in nature. For our discussions, the explicit forms of Q_r and $\tilde{Q}_b$ are not essential, but they can be found in [16, 17].

3 On-shell nilpotent BRST symmetries: augmented superfield approach

In this section, first of all, we take the chiral superfields, defined on the five (4,1)-dimensional super sub-manifold of

[4] We adopt here the conventions and notations such that the Minkowski 4D flat metric is $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$ on the spacetime manifold. The dot product $A \cdot B = A^a B^a$ and cross product $(A \times B)^a = f^{abc} A^b B^c$ are defined in the group space of the semi-simple Lie group. Here the Greek indices $\mu, \nu, \rho \ldots = 0, 1, 2, 3$ stand for the spacetime directions on the ordinary Minkowski spacetime manifold and the Latin indices $a, b, c \ldots = 1, 2, 3 \ldots$ denote the $SU(N)$ group indices in the "colour" space of the internal group manifold.

[5] The importance of the ghost fields appears in the proof of unitarity of a given physical process (allowed by the interacting non-Abelian gauge theory) where, for each gluon (a bosonic non-Abelian gauge field) Feynman loop diagram, a loop diagram constructed by the anti-commuting ghost field alone, is required (see, e.g., [18] for details).

the general six (4,2)-dimensional supermanifold and derive the on-shell nilpotent BRST symmetry transformations for the gauge- and the (anti-) ghost fields by exploiting the well-known horizontality condition. Later, in Sect. 3.2, we derive the nilpotent BRST transformations for the Dirac fields ψ and $\bar{\psi}$ by exploiting a gauge-invariant condition on the five (4,1)-dimensional chiral super sub-manifold.

3.1 Horizontality condition: on-shell nilpotent BRST symmetries for gauge and (anti-) ghost fields

For the present paper to be self-contained, we shall briefly invoke the essentials of the horizontality condition (connected with the usual superfield formalism [1–7]) in obtaining the on-shell nilpotent symmetry transformations (3) that exist for the non-Abelian gauge theory. For this purpose, we define the chiral (i.e., $\theta \to 0$) super exterior derivative and super 1-form connection, as taken from our earlier work (see, e.g. [20] for details):

$$\begin{aligned} \tilde{d}_{(c)} &= dx^\mu \partial_\mu + d\bar{\theta}\partial_{\bar{\theta}} , \\ \tilde{A}^{(1)}_{(c)} &= dx^\mu B^{(c)}_\mu(x,\bar{\theta}) + d\bar{\theta} F^{(c)}(x,\bar{\theta}) . \end{aligned} \tag{9}$$

The chiral super expansions for the following multiplet superfields are [20]

$$\begin{aligned} (B^{(c)}_\mu \cdot T)(x,\bar{\theta}) &= (A_\mu \cdot T)(x) + \bar{\theta}(R_\mu \cdot T)(x), \\ (F^{(c)} \cdot T)(x,\bar{\theta}) &= (C \cdot T)(x) + \mathrm{i}\bar{\theta}(B_1 \cdot T)(x) , \\ (\bar{F}^{(c)} \cdot T)(x,\bar{\theta}) &= (\bar{C} \cdot T)(x) + \mathrm{i}\bar{\theta}\,(B_2 \cdot T)(x) . \end{aligned} \tag{10}$$

In fact, these superfields are the chiral limits of the general super expansion that will be discussed in Sect. 4.1. In general, the super exterior derivative $\tilde{d} = dx^\mu\partial_\mu + d\theta\partial_\theta + d\bar{\theta}\partial_{\bar{\theta}}$ and the super 1-form connection $\tilde{A}^{(1)} = dZ^M A_M = dx^\mu B_\mu(x,\theta,\bar{\theta}) + d\theta \bar{F}(x,\theta,\bar{\theta}) + d\bar{\theta} F(x,\theta,\bar{\theta})$. It is evident that, in chiral limit ($\theta \to 0$), we have the definition of the super exterior derivative and super 1-form connection as given in (9). In the chiral expansion (10), $R_\mu \cdot T \equiv R^a_\mu T^a$, $B_1 \cdot T \equiv B^a_1 T^a$ and $B_2 \cdot T \equiv B^a_2 T^a$ are the group valued secondary fields. In fact, in the horizontality condition (i.e., $\tilde{F}^{(2)}_{(c)} = F^{(2)}$), we shall equate the super chiral 2-form $\tilde{F}^{(2)}_{(c)} = \tilde{d}_{(c)}\tilde{A}^{(1)}_{(c)} + i\tilde{A}^{(1)}_{(c)} \wedge \tilde{A}^{(1)}_{(c)}$, defined on the five (4,1)-dimensional super sub-manifold, where

$$\begin{aligned} \tilde{d}_{(c)}\tilde{A}^{(1)}_{(c)} &= (dx^\mu \wedge dx^\nu)\left(\partial_\mu B^{(c)}_\nu\right) + (dx^\mu \wedge d\bar{\theta}) \\ &\quad \times \left[\partial_\mu F^{(c)} - \partial_{\bar{\theta}} B^{(c)}_\mu\right] - (d\bar{\theta} \wedge d\bar{\theta})(\partial_{\bar{\theta}} F^{(c)}) , \end{aligned} \tag{11}$$

$$\begin{aligned} \mathrm{i}\tilde{A}^{(1)} \wedge \tilde{A}^{(1)} &= \mathrm{i}(dx^\mu \wedge dx^\nu)\left(B^{(c)}_\mu B^{(c)}_\nu\right) + \mathrm{i}(dx^\mu \wedge d\bar{\theta}) \\ &\quad \times \left[B^{(c)}_\mu, F^{(c)}\right] - \mathrm{i}(d\bar{\theta} \wedge d\bar{\theta})(F^{(c)} F^{(c)}) , \end{aligned} \tag{12}$$

with the ordinary 2-form ($F^{(2)} = dA^{(1)} + \mathrm{i}A^{(1)} \wedge A^{(1)} \equiv \frac{1}{2!}(dx^\mu \wedge dx^\nu)(F_{\mu\nu} \cdot T)$) curvature defined on the ordinary 4D spacetime manifold.

The consequences of the above horizontality condition $\tilde{F}^{(2)}_{(c)} = F^{(2)}$ on the five (4,1)-dimensional chiral super sub-manifold are

$$R_\mu = D_\mu C, \quad B_1 = -\frac{1}{2}(C \times C), \quad B_1 \times C = 0 . \tag{13}$$

The insertions of the above values into the expansion (10), vis-à-vis the on-shell nilpotent transformations (3), lead to

$$\begin{aligned} B^{(c)}_\mu(x,\bar{\theta}) &= A_\mu(x) + \bar{\theta}(\tilde{s}_b A_\mu(x)) , \\ F^{(c)}(x,\bar{\theta}) &= C(x) + \bar{\theta}(\tilde{s}_b C(x)) . \end{aligned} \tag{14}$$

It will be noted that the secondary field $(B_2 \cdot T)$ in the expansion of $\bar{F}^{(c)}$ is not determined by the above horizontality condition. The equation of motion $B + (\partial_\mu A^\mu) = 0$, however, comes to our rescue if we identify the secondary field B_2 with the Nakanishi–Lautrup auxiliary field B. In other words, we have the freedom to choose $B_2 \equiv B = -\left(\partial^\mu A_\mu\right)$. This ultimately leads to

$$\bar{F}^{(c)}(x,\bar{\theta}) = \bar{C}(x) + \bar{\theta}(\tilde{s}_b \bar{C}(x)) , \tag{15}$$

which is similar to the expansions of the other chiral superfields in (14).

We wrap up this subsection with the comment that the anti-chiral version of the above discussion does not lead to any appropriate nilpotent symmetry transformations (see, e.g. [20]). This provides, in a subtle way, the reason behind the non-existence of the on-shell nilpotent anti-BRST transformations for the above cited Lagrangian densities of the non-Abelian gauge theory. In contrast to this observation, for the Abelian $U(1)$ gauge theory, it has been shown [20] that the anti-chiral version of the above superfield formalism does lead to the derivation of the on-shell nilpotent (and anti-commuting) anti-BRST symmetry transformations.

3.2 Gauge-invariant condition: nilpotent BRST symmetry transformations for matter fields

To obtain the nilpotent BRST transformations for the matter fields $\psi(x), \bar{\psi}(x)$, we begin with the following gauge invariant condition on the chiral five (4,1)-dimensional super sub-manifold of the (4,2)-dimensional supermanifold:

$$\bar{\Psi}^{(c)}(x,\bar{\theta})(\tilde{d}_{(c)} + \mathrm{i}\tilde{A}^{(1)(h)}_{(c)})\Psi^{(c)}(x,\bar{\theta}) = \bar{\psi}(x)(d + \mathrm{i}A^{(1)})\psi(x) , \tag{16}$$

where the expansions for the chiral Dirac superfields are

$$\begin{aligned} \Psi^{(c)}(x,\bar{\theta}) &= \psi(x) + \mathrm{i}\bar{\theta}(b_1 \cdot T)(x) , \\ \bar{\Psi}^{(c)}(x,\bar{\theta}) &= \bar{\psi}(x) + \mathrm{i}\bar{\theta}(b_2 \cdot T)(x) . \end{aligned} \tag{17}$$

It is evident that the r.h.s. of (16) [i.e., $dx^\mu \bar{\psi}(x)(\partial_\mu + \mathrm{i}A_\mu)\psi(x)$] is a gauge (i.e., BRST) invariant quantity for the non-Abelian gauge theory described by the Lagrangian densities (1), (6) and (7). On the l.h.s. of (16), we have $\tilde{A}^{(1)(h)}_{(c)} = dx^\mu\left(A_\mu + \bar{\theta} D_\mu C\right) + d\bar{\theta}\left[C - \frac{i}{2}\bar{\theta}(C \times C)\right]$, which is the expression for the super 1-form $\tilde{A}^{(1)}_{(c)}$ after the application of the horizontality condition.

It is straightforward to note that the l.h.s. of (16) will produce the coefficients of the differentials $dx^\mu, dx^\mu(\bar{\theta}), d\bar{\theta}$ and $d\bar{\theta}(\bar{\theta})$. Out of which the coefficient of dx^μ will match with a similar kind of term emerging from the r.h.s. It is but natural that the rest of the coefficients are set equal to zero. For algebraic convenience, it is useful to first collect the coefficients of $d\bar{\theta}$ and $d\bar{\theta}(\bar{\theta})$ from the l.h.s. of (16). These coefficients are

$$- \mathrm{i} d\bar{\theta}[\bar{\psi} b_1 + \bar{\psi} C \psi] + d\bar{\theta}(\bar{\theta}) \times \left[b_2 b_1 + \frac{1}{2}\bar{\psi}(C \times C)\psi + \bar{\psi} C b_1 + b_2 C \psi \right] . \quad (18)$$

Setting the coefficients of $d\bar{\theta}$ and $d\bar{\theta}(\bar{\theta})$ equal to zero separately and independently, we obtain (for $\bar{\psi} \neq 0$), the following relationships

$$b_1 \equiv b_1 \cdot T = -(C \cdot T)\psi, \quad C b_1 + \frac{1}{2}(C \times C)\psi = 0 , \quad (19)$$

where, in the proof of the latter condition, the former condition $b_1 \cdot T = -\,(C \cdot T)\psi$ has been used as an input.

The explicit expressions for the coefficients of dx^μ and $dx^\mu(\bar{\theta})$, from the l.h.s. of (16), are collected together, as follows

$$dx^\mu(\bar{\psi} D_\mu \psi) - \mathrm{i} dx^\mu(\bar{\theta}) \left[\bar{\psi} D_\mu b_1 - b_2 D_\mu \psi + \bar{\psi} D_\mu C \psi \right] . \quad (20)$$

It is clear that the coefficient of dx^μ matches with that of the r.h.s. Using $b_1 = -(C \cdot T)\psi$, we obtain the following relationship

$$\mathrm{i} dx^\mu \bar{\theta}[\bar{\psi}(C \cdot T) + b_2] D_\mu \psi = 0 . \quad (21)$$

For our present interacting non-Abelian gauge theory, $D_\mu \psi \neq 0$. Thus, the solution that emerges from (21), is

$$b_2 \cdot T = -\bar{\psi}(C \cdot T) . \quad (22)$$

Substitution of b_1 and b_2 from (19) and (22) into (17) leads to the derivation of the nilpotent BRST symmetry transformations (3) for the matter fields, as given below

$$\Psi(x, \bar{\theta}) = \psi(x) + \bar{\theta}(\tilde{s}_b \psi), \quad \bar{\Psi}(x, \bar{\theta}) = \bar{\psi}(x) + \bar{\theta}(\tilde{s}_b \bar{\psi}) . \quad (23)$$

The above expansion, in terms of $\tilde{s}_b$, is exactly in the same form as the expansions (14) and (15). It should be emphasized that the gauge-covariant version of (16) (i.e., $(\tilde{d} + \mathrm{i}\tilde{A}^{(1)(h)})\Psi^{(c)} = (d + \mathrm{i}A^{(1)})\psi$) does not lead to the derivations of (23). Rather, it leads to an unphysical restriction: $D_\mu \psi = 0$.

The expansions in (14), (15) and (23) provide the geometrical interpretations for the on-shell nilpotent BRST symmetry $\tilde{s}_b$ (and the corresponding generator BRST charge $\tilde{Q}_b$) as the translation generator $(\partial/\partial\bar{\theta})$ along the Grassmannian direction $(\bar{\theta})$ of the chiral five (4,1)-dimensional super sub- manifold (parameterized by x^μ and $\bar{\theta}$) of the general six (4,2)-dimensional supermanifold. We note, in passing, that there is a mutual consistency and complementarity between the horizontality condition and the gauge invariant relation because (i) they are inter-linked, and (ii) they owe their origin to the super exterior derivatives $(\tilde{d})d$ and the super 1-form connection $(\tilde{A}^{(1)})A^{(1)}$.

4 Off-shell nilpotent symmetries: augmented superfield formalism

In this section, we shall derive the off-shell nilpotent symmetry transformations for all the fields of the (anti-) BRST invariant Lagrangian densities given in (1), (6), and (7) by applying the augmented superfield formalism.

4.1 Horizontality condition: off-shell nilpotent BRST and anti-BRST symmetries

For the paper to be self-contained, we recapitulate very briefly some of the key points of earlier work [3] on the horizontality condition in the context of non-Abelian gauge theory. We consider the superfields $B_\mu(x, \theta, \bar{\theta}), F\ (x, \theta, \bar{\theta})$ and $\bar{F}(x, \theta, \bar{\theta})$ that form the vector multiplet of the super 1-form connection $\tilde{A}^{(1)} = dZ^M A_M = dx^\mu B_\mu + d\theta \bar{F} + d\bar{\theta} F$ on the general six (4,2)-dimensional super-manifold. These component superfields can be expanded, in terms of the basic fields $A_\mu, C, \bar{C}$ and the secondary fields, along the Grassmannian directions of the above general supermanifold, as [3, 4, 11, 20]

$$\begin{aligned}(B_\mu \cdot T)(x, \theta, \bar{\theta}) &= (A_\mu \cdot T)(x) + \theta(\bar{R}_\mu \cdot T)(x) \\ &\quad + \bar{\theta}(R_\mu \cdot T)(x) + \mathrm{i}\theta\bar{\theta}(S_\mu \cdot T)(x) , \\ (F \cdot T)(x, \theta, \bar{\theta}) &= (C \cdot T)(x) + \mathrm{i}\theta(\bar{B}_1 \cdot T)(x) \\ &\quad + \mathrm{i}\bar{\theta}(B_1 \cdot T)(x) + \mathrm{i}\theta\bar{\theta}(s \cdot T)(x) , \\ (\bar{F} \cdot T)(x, \theta, \bar{\theta}) &= (\bar{C} \cdot T)(x) + \mathrm{i}(\bar{B}_2 \cdot T)(x) \\ &\quad + \mathrm{i}\bar{\theta}(B_2 \cdot T)(x) + \mathrm{i}\theta\bar{\theta}(\bar{s} \cdot T)(x) . \end{aligned} \quad (24)$$

It will be noted that the bosonic fields $A_\mu, S_\mu, B_1, \bar{B}_1, B_2, \bar{B}_2$ do match with the fermionic fields $R_\mu, \bar{R}_\mu, C, \bar{C}, s, \bar{s}$[6]. All the secondary fields will be expressed in terms of the basic and auxiliary fields of the above cited 4D (anti-) BRST invariant Lagrangian densities (6) and (7) for the non-Abelian gauge theory by tapping the potential of the horizontality condition on the general six (4,2)-dimensional supermanifold.

For the application of the horizontality condition in its full blaze of glory, it is important to define the super exterior derivative $\tilde{d} = dx^\mu \partial_\mu + d\theta \partial_\theta + d\bar{\theta} \partial_{\bar{\theta}}$ (with $\tilde{d}^2 = 0$) on the six (4,2)-dimensional supermanifold. Exploiting $\tilde{d}$ and $\tilde{A}^{(1)}$, one can define the super 2-form

[6] Hereafter, we shall be using, more often, the simpler notations $A_\mu \equiv A_\mu \cdot T$, $B_2 \equiv B_2 \cdot T$, etc., for the sake of brevity in the rest of the text.

curvature $\tilde{F}^{(2)} = \tilde{d}\tilde{A}^{(1)} + \mathrm{i}\tilde{A}^{(1)} \wedge \tilde{A}^{(1)}$ by exploiting the Maurer–Cartan equation. This is subsequently equated with the ordinary 2-form curvature $F^{(2)} = dA^{(1)} + \mathrm{i}A^{(1)} \wedge A^{(1)}$. This equality (the so-called horizontality condition) yields the following relationships, between the secondary fields on the one hand and the basic fields and auxiliary fields on the other hand, of the super expansion (24) (see, e.g., [3, 11, 20])

$$\begin{aligned} &R_\mu = D_\mu C, \quad \bar{R}_\mu = D_\mu \bar{C}, \quad \bar{B}_1 + B_2 = -(C \times \bar{C}) , \\ &\bar{s} = -\mathrm{i}(B_2 \times \bar{C}), \quad B_1 = -\frac{1}{2}(C \times C), \quad \bar{B}_2 = -\frac{1}{2}(\bar{C} \times \bar{C}) , \\ &s = \mathrm{i}(\bar{B}_1 \times C), \\ &S_\mu = D_\mu B_2 + D_\mu C \times \bar{C} \equiv -D_\mu \bar{B}_1 - D_\mu \bar{C} \times C . \end{aligned} \tag{25}$$

As we did earlier in Sect. 3, we identify $B_2 = B$ and $\bar{B}_1 = \bar{B}$ of the (anti-) BRST invariant Lagrangian densities (6) and (7) of Sect. 2. Having done this, we immediately obtain the (anti-) BRST transformations (2) and (5), because the super expansion in (24) can now be written, in terms of these transformations, as (see, e.g. [3, 11, 20] for details)

$$\begin{aligned} B_\mu^{(h)}(x, \theta, \bar{\theta}) &= A_\mu(x) + \theta(s_{ab}A_\mu(x)) + \bar{\theta}(s_b A_\mu(x)) \\ &\quad + \theta\bar{\theta}(s_b s_{ab} A_\mu(x)) , \\ F^{(h)}(x, \theta, \bar{\theta}) &= C(x) + \theta(s_{ab}C(x)) + \bar{\theta}(s_b C(x)) \\ &\quad + \theta\bar{\theta}(s_b s_{ab} C(x)) , \\ \bar{F}^{(h)}(x, \theta, \bar{\theta}) &= \bar{C}(x) + \theta(s_{ab}\bar{C}(x)) + \bar{\theta}(s_b \bar{C}(x)) \\ &\quad + \theta\bar{\theta}(s_b s_{ab} \bar{C}(x)) , \end{aligned} \tag{26}$$

where the superscript (h), on the above superfields, denotes the ensuing form of the superfields after the application of the horizontality condition. The above equation, vis-à-vis (8), provides the geometrical interpretation of the (anti-) BRST charges (that generate the off-shell nilpotent ($s_{(a)b}^2 = 0$) transformations $s_{(a)b}$) as the translational generators along the Grassmannian directions of the six (4,2)-dimensional supermanifold.

4.2 Gauge invariant condition: off-shell nilpotent (anti-) BRST symmetries for matter fields

To obtain the off-shell nilpotent BRST and anti-BRST symmetries together for the matter fields, we begin with the following gauge (i.e., BRST) invariant condition on the six (4,2)-dimensional supermanifold

$$\bar{\Psi}(x, \theta, \bar{\theta})(\tilde{d} + \mathrm{i}\tilde{A}^{(1)(h)})\Psi(x, \theta, \bar{\theta}) = \bar{\psi}(x) \left(d + \mathrm{i}A^{(1)}\right) \psi(x) , \tag{27}$$

where the superfields $\Psi(x, \theta, \bar{\theta})$ and $\bar{\Psi}(x, \theta, \bar{\theta})$, corresponding to the 4D spinors $\psi(x)$ and $\bar{\psi}(x)$, have the following expansions along the Grassmannian directions of the six (4,2)-dimensional supermanifold

$$\begin{aligned} \Psi(x, \theta, \bar{\theta}) &= \psi(x) + \mathrm{i}\theta(\bar{b}_1 \cdot T)(x) + \mathrm{i}\bar{\theta}(b_1 \cdot T)(x) \\ &\quad + \mathrm{i}\theta\bar{\theta}(f \cdot T)(x) , \end{aligned}$$

$$\begin{aligned} \bar{\Psi}(x, \theta, \bar{\theta}) &= \bar{\psi}(x) + \mathrm{i}\theta(\bar{b}_2 \cdot T)(x) + \mathrm{i}\bar{\theta}(b_2 \cdot T)(x) \\ &\quad + \mathrm{i}\theta\bar{\theta}(\bar{f} \cdot T)(x) , \end{aligned} \tag{28}$$

where (i) bosonic (commuting) fields $b_1, \bar{b}_1, b_2, \bar{b}_2$ match with the fermionic (anticommuting) fields $\psi, \bar{\psi}, f, \bar{f}$, (ii) the secondary fields $b_1, \bar{b}_1, b_2, \bar{b}_2, f, \bar{f}$ will be determined in terms of the basic and auxiliary fields of the Lagrangian densities (6) and (7), due to the gauge invariant restriction (27) invoked on the above general supermanifold, and (iii) the explicit form of the super 1-form connection $\tilde{A}^{(1)(h)}$, on the l.h.s. of (27), is: $\tilde{A}^{(1)(h)} = dx^\mu B_\mu^{(h)} + d\theta \bar{F}^{(h)} + d\bar{\theta} F^{(h)}$. The expressions for the redefined multiplet superfields $B_\mu^{(h)}, F^{(h)}, \bar{F}^{(h)}$ are given in (26).

It is clear that, on the r.h.s. of (27), we have the gauge (i.e., BRST) invariant quantity $(dx^\mu)[\bar{\psi}(x)(\partial_\mu + \mathrm{i}A_\mu \cdot T)\psi(x)]$ with a single differential dx^μ on the 4D spacetime sub-manifold of the six (4,2)-dimensional supermanifold. The l.h.s. will, however, produce the coefficients of the differentials $dx^\mu, d\theta$ and $d\bar{\theta}$. Out of these, only the coefficient of the pure dx^μ will match with the r.h.s. The rest of the coefficients of all the differentials will be set equal to zero. For algebraic convenience, it is helpful and handy to collect the coefficients of $d\theta$ and $d\bar{\theta}$ in the first go. The coefficients of $d\theta$ are

$$-\mathrm{i}d\theta[\bar{\psi}(\bar{b}_1 + \bar{C}\psi)] + d\theta(\theta)L_1 + d\theta(\bar{\theta})L_2 + d\theta(\theta\bar{\theta})L_3 , \tag{29}$$

where the explicit expressions for L_1, L_2 and L_3 are

$$\begin{aligned} L_1 &= \bar{b}_2\bar{b}_1 + \bar{b}_2\bar{C}\psi + \bar{\psi}\bar{C}\bar{b}_1 + \frac{1}{2}\bar{\psi}(\bar{C} \times \bar{C})\psi , \\ L_2 &= b_2\bar{b}_1 + \mathrm{i}\bar{\psi}f + b_2\bar{C}\psi - \bar{\psi}B\psi + \bar{\psi}\bar{C}b_1 , \\ L_3 &= \bar{f}\bar{b}_1 + \bar{b}_2 f + \bar{\psi}\bar{C}f + \bar{f}\bar{C}\psi - \mathrm{i}\bar{\psi}(B \times \bar{C})\psi - \mathrm{i}\bar{\psi}B\bar{b}_1 \\ &\quad - \frac{\mathrm{i}}{2}\bar{\psi}(\bar{C} \times \bar{C})b_1 + \mathrm{i}\bar{b}_2 B\psi - \mathrm{i}\bar{b}_2\bar{C}b_1 + \mathrm{i}b_2\bar{C}\bar{b}_1 \\ &\quad + \frac{\mathrm{i}}{2}b_2(\bar{C} \times \bar{C})\psi . \end{aligned} \tag{30}$$

Setting the coefficients of $d\theta, d\theta(\theta), d\theta(\bar{\theta})$ and $d\theta(\theta\bar{\theta})$ equal to zero, we obtain the following solutions (for $\bar{\psi} \neq 0$)

$$\bar{b}_1 = -(\bar{C} \cdot T)\psi, \quad f = -\mathrm{i}(B\psi - \bar{C}b_1) . \tag{31}$$

The rest of the conditions, which emerge after the imposition of the above restrictions, are satisfied if we use the above values of $\bar{b}_1$ and f.

In a similar fashion, collecting the coefficients of $d\bar{\theta}$ from the l.h.s. of (27), we obtain the following explicit expressions

$$-\mathrm{i}d\bar{\theta}[\bar{\psi}(b_1 + C\psi)] + d\bar{\theta}(\theta)M_1 + d\bar{\theta}(\bar{\theta})M_2 + d\bar{\theta}(\theta\bar{\theta})M_3 , \tag{32}$$

where the explicit forms of M_1, M_2 and M_3 are

$$\begin{aligned}
M_1 &= \bar{b}_2(b_1 + C\psi) + \bar{\psi}(C\bar{b}_1 - \mathrm{i}f - \bar{B}\psi) , \\
M_2 &= b_2(b_1 + C\psi) + \bar{\psi}[Cb_1 + \frac{1}{2}(C \times C)\psi] , \\
M_3 &= b_2[f + \mathrm{i}C\bar{b}_1 - \mathrm{i}\bar{B}\psi] + \bar{f}[b_1 + C\psi] \\
&\quad - \mathrm{i}\bar{b}_2 \left[Cb_1 + \frac{1}{2}(C \times C)\psi\right] , \\
&\quad + \bar{\psi}\left[Cf + \mathrm{i}(\bar{B} \times C)\psi + \frac{\mathrm{i}}{2}(C \times C)\bar{b}_1 + \mathrm{i}\bar{B}b_1\right] .
\end{aligned} \quad (33)$$

Setting the coefficients of $d\bar{\theta}$, $d\bar{\theta}(\theta)$, $d\bar{\theta}(\bar{\theta})$ and $d\bar{\theta}(\theta\bar{\theta})$ equal to zero separately and independently, we obtain the following solutions (for $\bar{\psi} \neq 0$)

$$b_1 = -(C \cdot T)\psi, \quad f = \mathrm{i}(\bar{B}\psi - C\bar{b}_1) . \quad (34)$$

It will be noted that the expressions for f, derived in (31) and (34), are consistent because, finally, they lead to $B + \bar{B} = -(C \times \bar{C})$, which has already been quoted in (25). The rest of the relations are found to be consistent if we use the values of $b_1, \bar{b}_1$ and f given in (31) and (34).

Finally, we collect the coefficients of dx^μ from the l.h.s. of (27). These are written, in their most explicit form, as follows

$$\begin{aligned}
& dx^\mu(\bar{\psi}\partial_\mu\psi + \mathrm{i}\bar{\psi}A_\mu\psi) \\
&\quad + \mathrm{i}dx^\mu(\theta)[\bar{b}_2 D_\mu\psi - \bar{\psi}D_\mu\bar{b}_1 - \bar{\psi}(D_\mu\bar{C})\psi] \\
&\quad + \mathrm{i}dx^\mu(\bar{\theta})[b_2 D_\mu\psi - \bar{\psi}D_\mu b_1 - \bar{\psi}(D_\mu C)\psi] \\
&\quad + \mathrm{i}dx^\mu(\theta\bar{\theta})\big[\bar{f}D_\mu\psi + \mathrm{i}\bar{b}_2(D_\mu b_1 + D_\mu C\psi) \\
&\quad - \mathrm{i}b_2(D_\mu\bar{b}_1 + D_\mu\bar{C}\psi) + \bar{\psi}\{D_\mu f - \mathrm{i}D_\mu\bar{C}b_1 \\
&\quad + \mathrm{i}D_\mu C\bar{b}_1 + \mathrm{i}D_\mu B\psi + \mathrm{i}(D_\mu C \times \bar{C})\psi\}\big] .
\end{aligned} \quad (35)$$

It is obvious that the coefficient of pure dx^μ matches with the corresponding coefficient emerging from the r.h.s. Exploiting the values of $\bar{b}_1, b_1$ and f from (31) and (34), and setting the coefficients of $dx^\mu(\theta), dx^\mu(\bar{\theta})$ and $dx^\mu(\theta\bar{\theta})$ equal to zero separately, leads to the following conditions

$$\begin{aligned}
& (\bar{b}_2 + \bar{\psi}\bar{C})D_\mu\psi = 0, \quad (b_2 + \bar{\psi}C)D_\mu\psi = 0 , \\
& \left[\bar{f} - \mathrm{i}\bar{b}_2 C + \mathrm{i}b_2\bar{C} - \mathrm{i}\bar{\psi}\left\{B + \frac{1}{2}(C \times \bar{C})\right\}\right] D_\mu\psi = 0 .
\end{aligned} \quad (36)$$

It is clear that, for the interacting non-Abelian gauge theory under consideration, we have $D_\mu\psi \neq 0$ because it leads to the existence of the interaction term in the theory. Thus, the solutions obtained from the above are

$$\begin{aligned}
\bar{b}_2 &= -\bar{\psi}\bar{C}, \quad b_2 = -\bar{\psi}C , \\
\bar{f} &= \mathrm{i}\bar{\psi}\left[B + \frac{1}{2}(C \times C)\right] .
\end{aligned} \quad (37)$$

Substitutions of the values from (31), (34) and (37) into the super expansions (28), lead to the following

$$\begin{aligned}
\Psi(x, \theta, \bar{\theta}) &= \psi(x) + \theta(s_{ab}\psi(x)) + \bar{\theta}(s_b\psi(x)) \\
&\quad + \theta\bar{\theta}(s_b s_{ab}\psi(x)) , \\
\bar{\Psi}(x, \theta, \bar{\theta}) &= \bar{\psi}(x) + \theta(s_{ab}\bar{\psi}(x)) + \bar{\theta}(s_b\bar{\psi}(x)) \\
&\quad + \theta\bar{\theta}(s_b s_{ab}\bar{\psi}(x)) ,
\end{aligned} \quad (38)$$

where the (anti-) BRST transformations $s_{(a)b}$ are illustrated in (2) and (5). Thus, we obtain the above nilpotent symmetry transformations for the matter fields together. The geometrical interpretations for the (anti-) BRST charges and the corresponding transformations $s_{(a)b}$ are same as the ones given for the gauge and (anti-) ghost fields in Sect. 3.

5 Conclusions

The derivation of the mathematically exact expressions for the nilpotent and anti-commuting (anti-) BRST symmetry transformations, associated with the matter fields of an interacting (non-) Abelian gauge theory, has been an outstanding problem within the framework of the superfield approach to BRST formalism. In our present endeavour, we have been able to resolve this long-standing problem because we have been able to derive the exact form of the nilpotent (anti-) BRST symmetry transformations for the matter (Dirac) fields of an $SU(N)$ non-Abelian gauge theory where there is an interaction between the non-Abelian gauge field and the matter (Dirac) fields[7]. In fact, the physical insights into the gauge (i.e., BRST) invariant quantities (cf. (16) and (27)) have helped us to obtain the proper restrictions on the five (4,1)-dimensional chiral super submanifold and the general six (4,2)-dimensional supermanifold that lead to the above exact derivations.

The above cited gauge (i.e., BRST) invariant quantities originate from the key properties associated with the (super) covariant derivatives on the supermanifold. Some of the striking similarities and key differences between the horizontality condition and the gauge invariant condition(s) are as follows. First, both primarily owe their origin to the (super) cohomological operators $\tilde{d}$ and d. Second, the geometrical origin and interpretations for the (anti-) BRST charges (and the nilpotent symmetry transformations they generate) remain intact for the validity of both conditions on the supermanifold. Third, whereas the horizontality condition is an $SU(N)$ covariant restriction (because $F^{(2)} \to UF^{(2)}U^{-1}$ where $U \in SU(N)$), the other condition(s), as the name suggests, is (are) the $SU(N)$ gauge invariant condition(s). Finally, as mentioned in the Sect. 3.2, the covariant versions of the gauge-invariant restrictions (cf. (16) and (27)) do not lead to the exact derivations of the nilpotent

[7] In fact, it is the conserved matter Noether current that couples to the gauge fields of the 1-form (non-) Abelian gauge theories to generate the interaction term when one requires the local gauge invariance in the theory (see, e.g. [21] for details).

symmetry transformations, whereas the horizontality condition (basically a covariant restriction) does lead to the exact derivations of the nilpotent symmetry transformations for the gauge and (anti-)ghost fields of the Lagrangian density of a non-Abelian gauge theory.

In our earlier work [8–13], we have consistently extended the horizontality condition by requiring the equality of the supersymmetric versions of the conserved currents/charges with the ordinary local conserved currents/ charges. In one of our recent papers [13], in addition to the horizontality condition, some suitable conserved quantities are required to be invariant on the supermanifold. In the present work and some of its precursors [14, 15], we have exploited the (super) covariant derivatives in the construction of the gauge (i.e., BRST) invariant quantities that have been equated on the specific supermanifolds. The latter yield the exact expressions for the nilpotent symmetries for the matter fields, whereas the former lead to the consistent derivations of the same. These consistent and complementary extensions of the horizontality condition have been christened as the augmented superfield approach to BRST formulation because they yield transformations for all the fields of a given 1-form gauge theory.

In his earlier work [22–27], one of the authors has studied in detail the gauge theories in a different type of superspace (also called the BRST superspace). The salient features of this work are that (i) the whole action including the source terms for the composite operators is accommodated in a single compact superspace action, (ii) the theory has a generalized gauge invariance in superspace, (iii) the superspace is completely unrestricted such that the operations like super rotation and translation, in anticommuting coordinates, can be carried out, and (iv) the WT identities are realized in a very simple manner [22–24]. These superspace formulations are used to study the renormalization of the gauge theories, in particular, the renormalization of gauge invariant operators [23–27]. It would be an interesting endeavour to study similar things in the present superfield formulation. Recently, the results of the covariant horizontality condition for Abelian $U(1)$ gauge theory have been derived from a gauge invariant condition involving covariant derivatives, Dirac fields and their connection with (super) 2-forms $(\tilde{F}^{(2)})F^{(2)}$ [28]. It would be a nice endeavour to check the same for the non-Abelian gauge theory. Furthermore, it would be very interesting to find a single restriction on the supermanifold that can produce the results of the horizontality condition and the new gauge invariant condition(s) together. These are some of the issues that are under investigation and our results will be reported elsewhere [29].

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